SYNOPSIS

The thesis submitted for Ph.D. Degree entitled **"PROPERTIES OF THE FAMILY OF GENERALIZED (a, b) – FIBONACCI NUMBERS".**

The subject matter of the topic is connected with the study of some aspects of Fibonacci numbers. The aim of the thesis is to study various properties of the family of generalized (a, b)-Fibonacci numbers. The thesis consists of six chapters. The **Zero Chapter** is reserved for the introduction to the chapters of the thesis.

The **First Chapter** consists of introduction of the sequence $\{\mathrm{F}_n^{\mathrm{G}}\}$, the sequence of *generalized Fibonacci numbers*. The terms of this sequence are defined by the recurrence relation

$$\mathrm{F}_n^{\mathrm{G}} = a^{\chi(n)} b^{1-\chi(n)} \mathrm{F}_{n-1}^{G} + c^{\chi(n)} d^{1-\chi(n)} \mathrm{F}_{n-2}^{\mathrm{G}}, \text{where } \chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases} \quad \textbf{(1)}$$

and a, b, c, d are any fixed real numbers.

Clearly for $a = b = c = d = 1$, the sequence $\{\mathrm{F}_{\mathrm{n}}^{\mathrm{G}}\}$ turns to be the sequence of usual Fibonacci numbers. In this thesis, we first develop extended Binet's formula for the various sub-sequences of $\{\mathrm{F}_n^{\mathrm{G}}\}$, which is obtained by considering some fixed values of a, b, c and d. We also derive some interesting identities and results for these subsequences.

We first consider the subsequence $\left\{F_n^{L(a,b)}\right\}$ of $\{\mathrm{F}_n^{\mathrm{G}}\}$ by considering $c = d = 1$ in (1). The terms of this subsequence are defined by the recurrence relation

$$F_n^{L(a,b)} = \begin{cases} aF_{n-1}^{L(a,b)} + F_{n-2}^{L(a,b)} \text{ ; when } n \text{ is odd} \\ bF_{n-1}^{L(a,b)} + F_{n-2}^{L(a,b)} \text{ ; when } n \text{ is even} \end{cases} (n \geq 2) \qquad \textbf{(2)}$$

with the initial conditions $F_0^{L(a,b)} = 0, F_1^{L(a,b)} = 1$.

For convenience, we write F_n^L for $F_n^{L(a,b)}$.

In this chapter, we first show that any two consecutive F_n^L's have no common factor, except 1. We then derive some simple identities for $\sum_{i=1}^{n} F_i^L$, $\sum_{i=1}^{n} F_{2i}^L$ and $\sum_{i=1}^{n} F_{2i}^L$. Using the techniques of generating functions, we next derive the explicit formula, known as extended Binet formula, for F_n^L as

$$F_n^L = \frac{b^{1-\chi(n)}}{(ab)^{[n/2]}}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right),$$

where α, β are the roots of the characteristic equation $x^2 - abx - ab = 0$ and $[x]$ denotes the integer part of x.

We then define $P_n = \alpha^n$, $Q_n = -\beta^n$ and derive the second order recurrence relation for P_n and Q_n. We also derive some simple and interesting properties involving α and β.

We next use the above extended Binet formula to derive an interesting combinatorial identity for F_n^L of the type

$$F_n^L = \frac{b^{\chi(n-1)}(ab)^{\lceil n/2 \rceil - \chi(n-1)}}{2^{n-1}} \left(\sum_{r=1}^{\infty} \binom{n}{2r-1} \left(1 + \frac{4}{ab}\right)^{r-1} \right).$$

We again use extended Binet's formula and find two different expressions for F_n^L in terms of F_{n-1}^L and some powers of α and β.

We then develop some nice results which are already published in the form of a research paper (See: Diwan, Shah [1]).

First of these results gives the following simple formula to obtain value of F_n^L for any positive integer *n*:

$$F_n^L = \left[\gamma \alpha^n + \frac{1}{2} \right],$$

where $\gamma = \frac{b^{1-\chi(n)}}{(ab)^{\lceil n/2 \rceil}(\alpha - \beta)}$ is a fixed constant.

We further derive recursive formula to compute the successor of any given generalized Fibonacci number. The formula reads as

$$F_n^L = \left[\frac{\alpha}{a^{1-\chi(n)} b^{\chi(n)}} F_{n-1}^L + \frac{1}{2} \right]; n \geq 2.$$

Finally we derive the reverse formula to compute the predecessor of a given generalized Fibonacci number as shown below:

$$F_n^L = \left[\frac{a^{\chi(n)} b^{1-\chi(n)}}{\alpha} \left(F_{n+1}^L + \frac{1}{2} \right) \right]; n \geq 2.$$

In **Second Chapter**, we continue our investigation and derive some more interesting properties related to the nth generalized Fibonacci number F_n^L.

We first derive *extended Catalen's identity* connecting three consecutive F_n^L's in arithmetic progressions to prove that

$a^{1-\chi(n-r)}b^{\chi(n-r)}F_{n-r}^{L}F_{n+r}^{L} - a^{1-\chi(n)}b^{\chi(n)}(F_n^L)^2 = b^{\chi(r)}a^{1-\chi(r)}(-1)^{n+1-r}(F_r^L)^2$,

for any two non-negative integers *n* & *r* with $n \geq r$.

We next prove the *extended d'Ocagne's identity* and show that

$a^{\chi(mn+n)}b^{\chi(mn+m)}F_m^L F_{n+1}^L - a^{\chi(mn+m)}b^{\chi(mn+n)}F_{m+1}^L F_n^L = (-1)^n b^{\chi(m-n)}F_{m-n}^L$,

where m, n are non-negative integers with $m \geq n$.

For any non-negative integer *n*, we also derive the following two different sums involving binomial coefficients:

a) $F_{2n}^L = \sum_{k=0}^{n}\binom{n}{k}b^{\chi(k)}(ab)^{\lceil k/2 \rceil}F_k^L$ b) b$F_{2n+1}^L = \sum_{k=0}^{n}\binom{n}{k}b^{\chi(k+1)}(ab)^{\lceil k+1/2 \rceil}F_{k+1}^L$.

We next derive exponential generating function for $\frac{F_{nk}^L}{n!}$ and $2^n\frac{F_n^L}{n!}$. Using the similar techniques of generating functions, we also prove the following identity:

$$F_{2n}^L = (ab)^n \sum_{k=0}^{n}\binom{n}{k}F_k^L .$$

Before the conclusion of this chapter, we derive the following two interesting results for the sequence of generalized Fibonacci numbers:

a) The number of digits of F_n^L is given by

$$\#\, F_n^L = [nlog\alpha + \log\mu] + 1, \text{ where } \mu = \frac{b^{1-\chi(n)}}{(ab)^{\lceil n/2 \rceil}(\alpha-\beta)} .$$

Here *log* represents the logarithm with base 10.

b) For any positive real number $m \geq 2$, we have $\sum_{i=1}^{\infty}\frac{F_i^L}{m^{(i+1)n}} = \frac{m^{2n}+bm^n-1}{m^{4n}-(ab+2)m^n+1}$.

The **Third chapter** deals with the study of generalized Lucas sequence $\left\{\mathrm{L}_n^{\mathrm{L(a,\,b)}}\right\}$ $(=\{L_n^L\})$ defined by the recurrence relation

$$L_n^L = a^{\chi(n)} b^{1-\chi(n)} L_{n-1}^L + L_{n-2}^L, \text{ with } L_0^L = 2, L_1^L = 1.$$

This can be equivalently expressed as $L_n^L = \begin{cases} aL_{n-1}^L + L_{n-2}^L \text{ ; if } n \text{ is odd} \\ bL_{n-1}^L + L_{n-2}^L \text{; if } n \text{ is even} \end{cases} (n \geq 2)$

with $L_0^L = 2, L_1^L = 1$.

We first derive simple identities for $\sum_{i=1}^n L_i^L$, $\sum_{i=1}^n L_{2i}^L$ and $\sum_{i=1}^n L_{2i+1}^L$. Next we derive the corresponding extended Binet's formula for L_n^L as

$$L_n^L = \frac{\gamma\alpha^n - \delta\beta^n}{(ab)^{[n/2]} b^{\chi(n)} (\alpha - \beta)},$$

where $\gamma = 2\alpha - 2ab + b$ and $\delta = 2\beta - 2ab + b$.

We further derive two relations connecting F_n^L and L_n^L. One of them reads as $L_n^L = \frac{\gamma}{n} F_n^L + \frac{2\beta^n}{(ab)^{[n/2]}\, b^{\chi(n)}}$.

We next derive the *extended Catalen's identity* for the three consecutive L_n^L's in arithmetic progression. We also derive ex*tended d' Ocagne identity* for L_n^L. We then find the two different expressions for L_n^L in terms of L_{n-1}^L along with the powers of α and β.

We also derive two sums involving binomial coefficients in the form $L_{2n}^L = \sum_{k=0}^n \binom{n}{k} b^{\chi(k)} (ab)^{\left[\frac{k}{2}\right]} L_k^L$ and $L_{2n+1}^L = \sum_{k=0}^n \binom{n}{k} b^{\chi(k+1)} (ab)^{\left[\frac{k+1}{2}\right]} L_{k+1}^L$.

We conclude the chapter by deriving the generating function for $\frac{L_n^L}{n!}$ and also by deriving an identity for L_{2n}^L.

In the **Fourth Chapter** we consider two subsequences $\left\{f_n^{L(a,b)}\right\}$ and $\left\{l_n^{L(a,b)}\right\}$ of $\{\mathrm{F_n^G}\}$ by considering $c = d = -1$ in (1).

In the first part of the chapter, we define terms of the subsequence $\left\{f_n^{L(a,b)}\right\}$ by the recurrence relation

$$f_n^{L(a,b)} = \begin{cases} af_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)}; \text{if } n \text{ is odd} \\ bf_{n-1}^{L(a,b)} - f_{n-2}^{L(a,b)}; \text{if } n \text{ is even} \end{cases} (n \geq 2) \qquad \textbf{(3)}$$

where $f_0^{L(a,b)} = 0, f_1^{L(a,b)} = 1$.

For convenience, we write f_n^L for $f_n^{\mathrm{L(a,b)}}$.

Using the techniques of generating functions, we derive the extended Binet formula for f_n^L as $f_n^L = \frac{b^{1-\chi(n)}}{(ab)^{[n/2]}}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$, where α, β are the roots of the characteristic equation $x^2 - abx + ab = 0$. We next derive some simple and interesting properties connecting α and β.

We then use the above extended Binet formula to derive an interesting combinatorial identity for f_n^L of the type

$$f_n^L = \frac{b^{\chi(n-1)}(ab)^{[n/2]-\chi(n-1)}}{2^{n-1}}\left(\sum_{r=1}^{\infty}\binom{n}{2r-1}\left(1+\frac{4}{ab}\right)^{r-1}\right).$$

We again use extended Binet's formula and find two different expressions for f_n^L in terms of f_{n-1}^L along with the powers of α and β. We also mention number of results which can be proved easily by the techniques used earlier in the thesis.

In the second part of the chapter, we study *generalized Lucas sequence* $\left\{l_n^{\mathrm{L(a,\,b)}}\right\}$ $(=\{l_n^L\})$ which is defined by the recurrence relation

$$l_n^L = a^{\chi(n)} b^{1-\chi(n)} l_{n-1}^L - l_{n-2}^L, \text{ with } l_0^L = 2, l_1^L = 1.$$

This can be equivalently expressed as

$$l_n^L = \begin{cases} a l_{n-1}^L - l_{n-2}^L \text{ ; if } n \text{ is odd} \\ b l_{n-1}^L - l_{n-2}^L \text{; if } n \text{ is even} \end{cases} (n \geq 2) \text{ with } l_0^L = 2, l_1^L = 1.$$

We derive the corresponding extended Binet's formula for $\{l_n^L\}$ as $l_n^L = \frac{\gamma\alpha^n - \delta\beta^n}{(ab)^{[n/2]} b^{\chi(n)}(\alpha-\beta)}$, where $\gamma = 2\alpha - 2ab + b$ and $\delta = 2\beta - 2ab + b$. We conclude the chapter by mentioning some interesting results which can be proved easily by the techniques used earlier.

In the **Fifth Chapter** we derive extended Binet's formula for nth term of some more different subsequences of $\{\mathrm{F_n^G}\}$. Two of them are already published in the form of research paper. (See: Diwan, Shah [2])

In this chapter we first consider the subsequence $\left\{F_n^{R(a,\,b)}\right\}$ of $\{\mathrm{F_n^G}\}$ by taking $a = b = 1$, $c = a, d = b$ in (1). The terms of this subsequence are defined by the recurrence relation

$$F_n^{R(a,b)} = \begin{cases} F_{n-1}^{R(a,b)} + aF_{n-2}^{R(a,b)}; \text{if } n \text{ is odd} \\ F_{n-1}^{R(a,b)} + bF_{n-2}^{R(a,b)}; \text{if } n \text{ is even} \end{cases} (n \geq 2) \text{ with } F_0^{R(a,b)} = 0, F_1^{R(a,b)} = 1.$$

For convenience, we write F_n^R for $F_n^{\mathrm{R(a,\,b)}}$.

Using the techniques of generating functions, we derive the extended Binet formula for F_n^R as

$$F_n^R = \frac{\left(\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - \delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}\right)}{\alpha-\beta}; \text{ where } \gamma = \alpha - b, \delta = \beta - b.$$

Here α and β are roots of equation $x^2 - (1 + a + b)x + ab = 0$. We derive some simple and interesting relations between α and β.

Next we consider the subsequence $\left\{L_n^{R(a,\,b)}\right\}$ of $\{\mathrm{F_n^G}\}$ defined by the recurrence relation

$$L_n^{R(a,b)} = \begin{cases} L_{n-1}^{R(a,b)} + aL_{n-2}^{R(a,b)}; \text{if } n \text{ is odd} \\ L_{n-1}^{R(a,b)} + bL_{n-2}^{R(a,b)}; \text{if } n \text{ is even} \end{cases} (n \geq 2) \text{ with } L_0^{R(a,b)} = 2, L_1^{R(a,b)} = 1.$$

We derive extended Binet's formula for this sequence.

We then reconsider the sequences $\left\{F_n^{L(a,\,b)}\right\}$ and $\left\{f_n^{L(a,\,b)}\right\} (= \{f_n^L\})$ defined in (2) and (3) respectively. In both the cases we consider the characteristic equation different from that considered in earlier chapters. By using the techniques of generating functions, we derive the different form of extended Binet formula for F_n^L and f_n^L as follows:

$$F_n^L = b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - \delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}\right); \text{ where } \gamma = \alpha - 1\,, \delta = \beta - 1$$

and

$$f_n^L = b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]}-\delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}\right); \text{ where } \gamma = 1+\alpha, \delta = 1+\beta.$$

In the first case, α, β are the roots of the characteristic equation $x^2-(ab+2)x+1=0$. Whereas, in the second case α, β are the roots of equation $x^2-(ab-2)x+1=0$. In both the cases, we derive some curious properties relating α and β.

In the final and **Sixth Chapter**, we consider the subsequences $\left\{H_n^{L(a,b)}\right\}$ and $\left\{H_n^{R(a,b)}\right\}$ of $\{\mathrm{F_n^G}\}$ by considering $b=c=1$ and $d=b$ in (1) for first case, and by considering $a=d=1$ and $c=a$ in (1) for second case. The terms of this subsequences are defined respectively by the following recurrence relations:

a) $H_n^{L(a,b)} = \begin{cases} aH_{n-1}^{L(a,b)} + H_{n-2}^{L(a,b)}; \text{if } n \text{ is odd} \\ H_{n-1}^{L(a,b)} + bH_{n-2}^{L(a,b)}; \text{if } n \text{ is even} \end{cases} (n \geq 2)$

with the initial condition $H_0^{L(a,b)} = 0, H_1^{L(a,b)} = 1$.

b) $\mathrm{H_n^{R(a,b)}} = \begin{cases} H_{n-1}^{R(a,b)} + aH_{n-2}^{R(a,b)}; \text{if } n \text{ is odd} \\ bH_{n-1}^{R(a,b)} + H_{n-2}^{R(a,b)}; \text{if } n \text{ is even} \end{cases} (n \geq 2)$

with the initial condition $H_0^{R(a,b)} = 0, H_1^{R(a,b)} = 1$.

For convenience, we write H_n^L and H_n^R for $H_{\mathrm{n}}^{\mathrm{L(a,b)}}$ and $H_{\mathrm{n}}^{\mathrm{R(a,b)}}$ respectively.

Then by using the techniques of generating functions, we derive the extended Binet formula for H_n^L and H_n^R as follows:

$$H_n^L = \frac{\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - \delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}, \text{ where } \gamma = \alpha - b, \delta = \beta - b,$$

and α, β are the roots of the characteristic equation $x^2 - (a+b+1)x + b = 0$;

$$H_n^R = b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - \delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}\right), \text{ where } \gamma = \alpha - 1, \delta = \beta - 1,$$

and α, β are the roots of the characteristic equation $x^2 - (a+b+1)x + a = 0$.

Both these results are accepted for publication in the research journal. (See: Diwan, Shah [3])

We also derive some simple properties relating α and β in both the cases.

At the end of thesis we provide appendices which give first few terms of all the subsequences of $\{F_n^G\}$ considered in the thesis.

References:

[1] Diwan Daksha M., Shah Devbhadra V.: *Explicit and recursive formula for class of generalized Fibonacci sequence*, Proceedings of 19th Annual Cum 4th International Conference of GAMS, 2014, 104 – 108. [ISBN: 978-81-928100-5-8]

[2] Diwan Daksha M., Shah Devbhadra V.: *Extended Binet formula for the class of generalized Fibonacci sequences*, Proceedings of 19th Annual Cum 4th International Conference of GAMS, 2014, 109 – 113. [ISBN: 978-81-928100-5-8]

[3] Diwan Daksha M., Shah Devbhadra V.: *Extended Binet's formula for the class of generalized Fibonacci sequences*, VNSGU Journal of Science and Technology, 2015. (To be appeared)